Multiple Solutions for a Classical Problem in the Calculus of Variations

I. Ekeland

Université Paris—Dauphine, CEREMADE, Paris, France

and

N. Ghoussoub and H. Tehrani*

Department of Mathematics, The University of British Columbia, Vancouver, British Columbia V6T 1W2, Canada

Received December 11, 1995; revised June 10, 1996

INTRODUCTION

The most classical problem in the calculus of variations consists of seeking a curve with given end points minimizing a given integral

\[ \inf_{\gamma} \int_0^T L(t, x, \dot{x}) \, dt \]  

over all suitable paths satisfying

\[ x(0) = x_0, \quad x(T) = x_1 \quad \text{in} \quad \mathbb{R}^N \]

There are countless references for this problem. It was solved in full generality by Tonelli, who invented lower semi-continuity for that purpose. He was fortunate in working after Lebesgue had redefined the notion of integral, so that he could use Lebesgue's theory to prove that the integrals arising in the classical calculus of variations were lower semi-continuous.

We refer to the treatise of Tonelli [To] for a history of the subject until 1923, and to the recent book by Giusti [Gi] for a beautiful and concise description of the state of the art today (including its extension to several variables). Of course, there are many other books on the subject, some of which have been extremely popular: Bolza [Bo], Bliss [Bl], Caratheodory [C], Gelfand–Fomin [G–F]. The book by Bolza, in fact, became so popular that the fixed-endpoint problem we stated became known as the
“Bolza problem”, although it goes back to the last years of the seventeenth century. In his “Principia” of 1686, Newton found the shape of a shell which would minimize air resistance, and in 1696, Jakob Bernoulli found the curve that would bring a heavy body from A to B in the least time (A is higher than B, and the body slides along the curve with zero initial velocity). Both are fixed-endpoint problems of type (*).

However, all these works deal with the case when the integrand \( L \) is coercive, that is
\[
L(t, x, y) \geq a + b |y|^p
\]
for some constant \( p > 1 \). This will imply that the integral is bounded from below, and the infimum is attained. Little or no attention has been paid to the case when \( L \) is not coercive, so that the integral is unbounded from above and below. Typical of this situation is:
\[
L(t, x, y) = \frac{1}{2} |y|^2 - \frac{1}{p} |x|^p
\]
In such a case, there can be no question of minimizing the integral, so we have to settle for critical points, i.e. solutions of the corresponding Euler equation
\[
\ddot{x} + x |x|^{p-2} = 0
\]
satisfying the boundary conditions
\[
x(0) = x_0, \quad x(T) = x_1 \quad \text{in} \ \mathbb{R}^n.
\]
In that case, equation (3) can be solved explicitly by quadratures (see Arnold [A], Chapt. 2, Sec 8): all orbits are planar, and if \( p \neq 2 \) they are dense in an annulus (the higher the energy, the wider the annulus). It follows that, if \( p > 2 \) problem (3)-(4) has infinitely many solutions, with higher and higher energy.

It is the purpose of this paper to investigate more general boundary-value problems, which are not explicitly solvable, namely,
\[
\ddot{x} + V'(x) = 0
\]
\[
x(0) = x_0, \quad x(T) = x_1,
\]
where \( V(x) \sim |x|^p \) at infinity. It will be shown that, if the potential is even, and \( 2 < p < 4 \), this problem has infinitely many solutions, with higher and higher energy.

The method of the proof, of course, is quite different from the ones used in classical calculus of variations. It is no longer a question of minimizing
the corresponding integral, one must find a critical point. Ways to do this
have been developed in the study of periodic solutions for equations of type
(5), following the pioneering work of Rabinowitz [R]. But these methods
rely on an invariance property, namely that if \(x(t)\) is a solution, so is
\(x(t + \theta)\) for any \(\theta\). This is certainly true for a periodic solution of (5), but
not for a solution of the boundary-value problem (5)–(6). We do, however,
rely on these methods, by means of perturbation results developed by
Bahri–Berestycki [Ba–Be], Struwe [St] and Rabinowitz [R].

Here is the precise statement of our main result:

**Theorem 1.** Suppose \(V \in C^1(\mathbb{R}^N, \mathbb{R})\) is even and for some \(2 < p < 4\),
satisfies the following conditions:

(i) \(0 < pV(x) \leq \langle V'(x), x \rangle\) for all \(|x|\) large.

(ii) \(|V'(x)| \leq \alpha |x|^{p-1} + \beta\) for all \(x\), where \(\alpha, \beta\) are positive scalars.

Then for any \(x_0, x_1 \in \mathbb{R}^N\) and \(T > 0\), the following Bolza problem

\[
\ddot{x} + V'(x) = 0 \quad x(0) = x_0 \quad \text{and} \quad x(T) = x_1
\]

has infinitely many solutions.

The case \(1 < p < 2\) has been partially treated (for convex potentials) by
Clarke and Ekeland [C–E], where they find one solution under some
restriction on the time interval \(T\). We do not know what happens if \(p \geq 4\)
as the “perturbation from symmetry proof” breaks down beyond that value
of \(p\), but we have no counterexample to existence.

In an earlier version of this paper, the above result was proved for
\(2 < p < 1 + \sqrt{5}\) and under an additional condition on (the second
derivative) of \(V\). The improvement in \(p\) is obtained by exploiting the information
on the Morse indices, while the condition on \(V''\) was eliminated by
using Rabinowitz’ method as opposed to Struwe’s. We are thankful to G.
Fang for pointing out to us the work of Tanaka [Ta] and Berestycki [Be].

**The Proof.** First, we briefly sketch the idea of the proof. Take two
points \(x_0\) and \(x_1\) in \(\mathbb{R}^N\) and \(T > 0\), we are interested in the boundary value problem

\[
\begin{aligned}
\ddot{x} + V'(x) &= 0 \\
x(0) &= x_0, \quad x(T) = x_1,
\end{aligned}
\]

where \(V \in C^1(\mathbb{R}^N, \mathbb{R})\) is even and satisfies suitable growth conditions for \(|x|\)
large. Consider the following “change of variable”: \(z: [0, T] \to \mathbb{R}^N\).
z(t) = x_0 + \left( x_1 - x_0/T \right) \cdot t, \text{ and } u = x - z. \text{ Then, the boundary value problem (I) can be rewritten in the form:}

\begin{align*}
\begin{cases}
\ddot{u} + V'(u + z(t)) = 0 \\
u(0) = 0, \quad u(T) = 0.
\end{cases}
\end{align*}

Solutions of (II) are critical points of the “action functional”

\[ I(u) = \int_0^T \left( \frac{\dot{u}^2}{2} - V(u + z(t)) + V(z(t)) \right) dt \]

defined on the Sobolev space \( H = H^1_0(0, T; \mathbb{R}^N) \). We use the method of perturbation from symmetry as developed by Bahri–Berestycki [Ba–Be], Struwe [S] and Rabinowitz [R]. To this end, we first consider the associated even functional \( \bar{I} \)

\[ \bar{I}(u) = \int_0^T \left( \frac{\dot{u}^2}{2} - V(u) \right) dt. \]

Applying the methods of Bahri–Berestycki [Ba–Be] and Tanaka [Ta], we prove the existence of a sequence of critical points of \( \bar{I} \) and derive an estimate from below on the growth of the associated critical values which are defined as minimax values over suitable sets

\[ e_n = \inf_{h \in J_n} \sup_{u \in V_n} \bar{I}(h(u)). \]

Next we apply the method of Rabinowitz [R] to \( I \) and obtain the existence of infinitely many solutions to (II) under some assumptions on the growth of minimax values

\[ d_n = \inf_{h \in J_n} \sup_{u \in V_n} I(h(u)). \]

Finally a comparison argument involving \( I \) and \( \bar{I} \) provides the necessary growth estimate on \( d_n \) and completes the proof of Theorem 1.

Now a few words about the notation. Throughout this paper, we use \( \langle \cdot, \cdot \rangle \) to denote inner product of vectors in \( \mathbb{R}^N \), and \( | \cdot | \) the associated norm. The norm in the space \( H = H^1_0(0, T; \mathbb{R}^N) \) is denoted by \( \| \cdot \| \), while \( \| \cdot \|_P \) is used for the norm in \( L^p(0, T; \mathbb{R}^N) \). Finally as is clear from (1), without loss of generality we may take \( V(0) = 0 \), which is assumed throughout.
Lemma 2. Suppose $V \in C^1(\mathbb{R}^N, \mathbb{R})$ is even and satisfies for some $p$ and $q$, $2 < p \leq q$ and $r_0 > 0$ the following conditions:

$$0 < p V(x) \leq \langle V'(x), x \rangle \quad \text{for} \quad |x| \geq r_0 \quad (7)$$

$$V(x) \leq a_1 |x|^q + a_2 \quad a_1, a_2 > 0. \quad (8)$$

Then, the (even) functional

$$I(u) = \int_0^T \left( \frac{1}{2} \dot{u}^2 - V(u) \right) dt \quad u \in H,$$

has a sequence $(c_n)_{n \geq 1}$ of critical values that satisfy for some constant $C > 0$,

$$c_n \geq C a_1^{q/p - 2}.$$

Proof. To define the sequence $(c_n)_{n \geq 1}$ we use a critical point theorem for even functionals due to Ambrosetti and Rabinowitz [A–R]. For the general setting we consider a Hilbert space $E$ and a functional $J \in C^1(E, \mathbb{R})$ which satisfies the Palais–Smale compactness condition (P–S): Whenever a sequence $(v_n)_{n=1}^\infty$ in $E$ satisfies for some $M > 0$,

$$I(v_n) \leq M \quad \text{for all} \quad n,$$

$$\Gamma(v_n) \to 0 \quad \text{in} \quad E^* \quad \text{as} \quad n \to \infty$$

there is a subsequence of $(v_n)$ which converges in $E$. Here we state

Theorem 3 ([A–R]). Suppose $J$ is even, that is $J(u) = J(-u)$ and let $E^+$ and $E^-$ be closed subspaces of $E$ with $\dim E^- - \text{codim} E^+ = 1$. Suppose

(i) $J(0) = 0$

(ii) $\exists \alpha > 0, \rho > 0$ such that $J(u) \geq \alpha$ for $u \in S_\rho^+ = \{u \in E^+: \|u\| = \rho\}$

(iii) $\exists R > 0$ such that $u \in E^-$ and $\|u\| \geq R \Rightarrow J(u) \leq 0$.

Consider the following class of maps:

$$\Gamma = \{ h \in C^0(E, E); h \text{ is odd}, h(u) = u \text{ if } u \in E^-, \|u\| \geq R \}.$$

Then

(a) For all $\delta > 0$ and $h \in \Gamma$, we have that $S_\delta^+ \cap h(E^-) \neq \emptyset$

(b) The number

$$\beta = \inf_{h \in \Gamma} \sup_{u \in E^-} J(h(u)) \geq \alpha$$

is a critical value for $J$. 

SOLUTIONS FOR A CLASSICAL PROBLEM 233
We apply this theorem repeatedly to define \((c_n)_{n \geq 1}\). For that, let \(e_1, e_2, \ldots, e_N\) denote the usual basis of \(\mathbb{R}^N\). The operator \(S(u) = -\dot{u}\) defined on the space \(W = H_0^1(0, T; \mathbb{R}^N) \cap H^2(0, T; \mathbb{R}^N)\) has as its eigenvalues the numbers \(\{(\pi/T)^2 k^2 : k = 1, 2, 3, \ldots\}\). Note that each eigenvalue has multiplicity \(N\). In fact the eigenspace associated with the eigenvalue \((\pi/T)^2 k^2\) is the span of \(\{\sin((\pi/T) j t) e_j : 1 \leq j \leq N\}\). So let \(\{\lambda_k : k \geq 1\}\) be the set of all eigenvalues of \(S\) (each eigenvalue repeated \(N\) times) and let \(H^m (m \geq 2)\) be the eigenspace associated with the first \(m\) eigenvalues. Set

\[
E_0^- = H^m, \quad E_0^+ = (H^{m-1})^\perp
\]

(orthogonal complement of \(E_{m-1}\) in \(H\)),

where \(m_0\) is the smallest integer greater than or equal to \(N\) such that

\[
\frac{\sqrt{\pi} T + 1}{2 \pi m_0/N} \leq \frac{1}{\sqrt{2}}.
\]

Next we show that \(I\) satisfies the conditions of Theorem 3. These conditions are standard. In fact by (8), \(I\) defines an even \(C^1\) function on \(H\), and (7) implies the existence of positive constants \(a_3, a_4, a_5\) such that:

\[
\frac{1}{p} \left[ (V'(x), x) + \lambda_1 \right] \geq V(x) + a_4 \geq a_5 |x|^p
\]

which in turn implies that for any finite dimensional subspace \(F\) of \(H\), there exists \(R_F > 0\) such that \(I(u) \leq 0\) for \(\|u\| \geq R_F\), \(u \in F\). Using (10) once more one can show that \(I\) satisfies the \(P-S\) condition. Finally, by using the min-max characterization of the eigenvalues

\[
\lambda_k = \inf_{u \in (H^i)^\perp} \frac{\int_0^T |u|^2 dt}{\int_0^T \frac{1}{u} dt} \quad (H^0 = H)
\]

we derive

\[
\begin{align*}
\bar{I} &\geq \int_0^T \frac{\dot{u}^2}{2} dt - a_1 \int_0^T |u|^q dt - a_2 T \\
&\geq \int_0^T \frac{\dot{u}^2}{2} dt - a_1 \int_0^T |u|^q dt - a_2 T \\
&\geq \int_0^T \frac{\dot{u}^2}{2} dt - a_1 |T|^{q-2/2} \|u\|^{q-2} \int_0^T |u|^2 dt - a_2 T \\
&\geq \left( \frac{1}{2} \frac{a_1 T^{q-2/2}}{(\pi/T)^2} \frac{|m_0/N|^2}{\|u\|^{q-2}} \right) \|u\|^2 - a_2 T \\
&\geq 1
\end{align*}
\]

for \(u \in E_0^+\)
for $u \in S_{\rho}^{n} = \{ u \in E_{n} : \|u\| = \rho \}$, \( \rho = 2(a_{2}T + 1)^{1/2} \) by (9). So Theorem 3 applies and we get a critical value

$$c_{0} = \inf_{h \in \Gamma_{0}} \sup_{u \in E_{0}} \bar{I}(h(u))$$

with

$$\Gamma_{0} = \{ h \in C^{0}(H, H) : h \text{ is odd, } h(u) = u \text{ if } u \in E_{0}^{-}, \|u\| \geq R_{0} \}.$$ Following this procedure, we define $c_{n}, n \geq 1$, by working with

$$E_{n}^{-} = H^{m+1} \quad E_{n}^{+} = (H^{m+n-1})^{+}$$

$$\Gamma_{n} = \{ h \in C^{0}(H, H) : h \text{ is odd, } h(u) = u \text{ if } u \in E_{n}^{-}; \|u\| \geq R_{n} \},$$

where

$$R_{n} = R_{E_{n}^{-}} \quad \text{and} \quad R_{n} > R_{n+1}$$

and

$$c_{n} := \inf_{h \in \Gamma_{n}} \sup_{u \in E_{n}} \bar{I}(h(u)).$$

Note that if we define $D_{n} = \{ x \in E_{n}^{-} : \|x\| \leq R_{n} \}$, then (11) shows that

$$c_{n} = \inf_{h \in \Gamma_{n}} \sup_{u \in D_{n}} \bar{I}(h(u)). \quad (12)$$

To derive the growth estimate of the lemma for $(c_{n})_{n \geq 0}$, we use ideas of Bahri-Berestycki [Ba–Be] and Tanaka [Ta]. First using (10) we have:

$$\bar{I}(u) \geq \int_{0}^{T} \left( \frac{|u'|^{2}}{2} - a_{1} |u|^{q} \right) dt - a_{2}T$$

$$\geq \sum_{i=1}^{N} \int_{0}^{T} \left( \frac{|u|^{2}}{2} - a_{1} 2^{qN_{i}} |u'|^{q} \right) dt - C \quad u = (u_{1}, ..., u^{N})$$

so if we define $K(u) : H \to R$ by $K(u) = \sum_{i=1}^{N} \left[ \int_{0}^{T} \left(|u'|^{2}/2 \right) - (\sigma_{i}/q) |u'|^{q} \right] dt$, then:

$$c_{n} = \inf_{h \in \Gamma_{n}} \sup_{u \in D_{n}} \bar{I}(h(u)) \geq d_{n} := \inf_{h \in \Gamma_{n}} \sup_{u \in D_{n}} K(h(u)) - C.$$ The same analysis as above implies that $(d_{n})_{n \geq 0}$ is a sequence of critical values of $K$. Furthermore, we apply Theorem (A) in Tanaka [Ta] to get a sequence $(v_{n})_{n \geq 0}$ in $H$ such that
where

\[ \text{index}_0 K^*(v_n) = \max \{ \dim E, E \subseteq H \} \text{ is a subspace such that} \]
\[ K^*(v_n) \leq 0 \text{ for } e \in E \} \]

Writing each \( v_n \) as \( v_n = (v_1^n, v_2^n, \ldots, v_N^n) \) and considering the form of the functional \( K \), we get that each \( v_i^n \), \( 1 \leq i \leq N \), is a solution of the following 1-dimensional ODE.

\[ -\ddot{x} = \alpha_1 |x|^{q-2} x \quad \text{in } (0, T) \quad (16) \]
\[ x(0) = x(T) = 0 \quad (17) \]
\[ d_n = d_1^n + \cdots + d_N^n \quad (18) \]
\[ \text{index}_0 v_n = \sum_{i=1}^{N} \text{index}_0 v_i^n, \quad (19) \]

where \( d_i^n \) and \( \text{index}_0 v_i^n \) are respectively a critical value and the augmented Morse index of the corresponding critical point \( v_i^n \) of the one-dimensional functional \( L \) on \( H^1_0(0, T, \mathbb{R}^3) \) defined by

\[ L(u) = \int_0^T \left( \frac{1}{2} |\dot{u}|^2 + \Psi(u) (V(u + z) - V(u)) \right) dt \quad (21) \]

This completes the proof of lemma 2.
where $\mathcal{P}(u) = \Phi(Q^{-1}(u) \int_0^T (V(u+z) + a_4) \, dt)$, $a_4$ is as defined in (10), $Q(u) = 2\mathcal{A}(I(u)^2 + 1)^{1/2}$ with $A$ a suitably defined constant and $\Phi$ is a function in $C^\infty(\mathbb{R}, [0, 1])$ which is equal to 1 on $(-\infty, 1]$, to 0 on $[2, +\infty)$ and such that $\Phi'(t) \in (-2, 0)$ for $t \in (1, 2)$.

To motivate this we now assume:

$$|V'(x)| \leq \alpha_1 |x|^p - \alpha_2; \quad \alpha_1, \alpha_2 > 0. \quad (22)$$

Then if $u$ is a critical point of $I$, simple estimates yield

$$\left| \int_0^T V(z(t)) \, dt + I(u) \right| = I(u) - \frac{1}{2} \left\langle I'(u), u \right\rangle$$

$$= \int_0^T \frac{1}{2} \left( V'(u+z), u \right) - V(u+z) \right) \, dt$$

$$\geq \left( \frac{1}{2} - \frac{1}{p} \right) \int_0^T \left[ \left( V'(u+z), u \right) + a_3 \right] \, dt$$

$$- \int_0^T \frac{1}{2} \left( V'(u+z), z \right) \, dt$$

$$\geq \left( \frac{p}{2} - 1 \right) \int_0^T \left[ V(u+z) + a_4 \right] \, dt$$

$$- \alpha_6 \int_0^T |u+z|^q \, dt$$

$$\geq \left( \frac{p}{2} - 1 \right) \int_0^T \left[ V(u+z) + a_4 \right] \, dt$$

$$- \alpha_7 \int_0^T |u|^q \, dt - \alpha_8$$

$$\geq \left( \frac{p}{2} - 1 \right) \int_0^T \left[ V(u+z) + a_4 \right] \, dt$$

$$- \frac{p-2}{4\alpha_5} \int_0^T |u|^q \, dt - \alpha_9$$

$$\geq \frac{p-2}{4} \int_0^T \left[ V(u+z) + a_4 \right] \, dt - \alpha_{10} \quad (23)$$

which implies

$$\int_0^T \left[ V(u+z) + a_4 \right] \, dt \leq A(I(u)^2 + 1)^{1/2}.$$

The reason for the introduction of $J$ is part 20 of the following theorem:
Theorem 4 (see Proposition 10.16 in [R]). Suppose that \( V \in C^1(\mathbb{R}^N, \mathbb{R}) \) also satisfies conditions (7) and (21), that is:

\[
0 < pV(x) \leq \langle V'(x), x \rangle \quad \text{for} \quad |x| \geq r_0
\]

\[
|V'(x)| \leq \alpha_1 |x|^{\alpha - 1} + \alpha_2 \quad \alpha_1, \alpha_2 > 0.
\]

Then,

1. \( J \in C^1(\mathbb{H}, \mathbb{R}) \).
2. There exists a constant \( \beta_1 \) (independent of \( u \)) such that

\[
|J(u) - J(-u)| \leq \beta_1(|J(u)|^{\alpha - 1} + 1)
\]

3. There is a constant \( M_0 > 0 \) such that if \( J(u) \geq M_0 \) and \( J'(u) = 0 \), then \( J(u) = R(u) \) and \( I'(u) = 0 \).

4. \( J \) satisfies \( (P-S)_{\infty} \) at \( e \), for \( e \) large enough.

Proof. 1. is trivial. For (2), use first (21) to get that

\[
\Psi(u) \int_0^T |V(u + z) - V(u)| \, dt \leq \alpha_1(|J(u)|^{\alpha - 1} + 1) \quad \alpha_1 \text{ independent of } u.
\]

and then use (22).

To prove 3 we calculate

\[
\langle J'(u), u \rangle = \int_0^T (|u|^2 - \langle V'(u), u \rangle ) \, dt - \int_0^T (V(u + z) - V(u)) \, dt
\]

\[
- \Psi(u) \left[ \int_0^T \langle V'(u + z), u \rangle \, dt - \int_0^T \langle V'(u), u \rangle \, dt \right]
\]

and

\[
\langle \Psi'(u), u \rangle = \Phi'(\theta(u)) Q(u)^{-2} \int_0^T Q(u)^{-1} \langle V'(u + z), u \rangle \, dt
\]

\[
- (2A)^2 \theta(u) I(u) \langle I'(u), u \rangle
\]

\[
\theta(u) = Q(u)^{-1} \int_0^T (V(u + z) + a_1) \, dt.
\]
Regrouping we have
\[
\langle J'(u), u \rangle = \left[ 1 + T_1(u) \right] \int_0^T |\dot{u}|^2 \, dt - [ \Psi(u) + T_2(u) ]
\times \int_0^T \langle V'(u + z), u \rangle \, dt + (\Psi(u) - 1) \int_0^T \langle V'(u), u \rangle \, dt,
\]
where
\[
T_1(u) \equiv \Phi'(\theta(u)) \frac{Q(u)}{2} (2A)^2 f(u) \theta(u) \cdot \left( \int_0^T (V(u + z) - V(u)) \, dt \right)
\]
\[
T_2(u) \equiv \Phi'(\theta(u)) \frac{Q(u)}{2} \left( \int_0^T \langle V'(u + z), u \rangle \, dt \right)
\times \left( \int_0^T (V(u + z) - V(u)) \, dt \right) + T_1(u).
\]
Now
\[
I(u) = \frac{\langle J'(u), u \rangle}{2(T_1(u))} = \frac{1}{2} \left( \frac{\Psi(u) + T_2(u)}{1 + T_1(u)} \right) \int_0^T \langle V'(u + z), u \rangle \, dt
\]
\[ - \int_0^T V(u + z) \, dt + \frac{1}{2} \left( \frac{1 - \Psi(u)}{1 + T_1(u)} \right) \int_0^T \langle V'(u), u \rangle \, dt
\]
\[
= \frac{\Psi(u) + T_2(u)}{1 + T_1(u)} \times \left[ \int_0^T \left( \frac{1}{2} \langle V'(u + z), u \rangle - V(u + z) \right) \, dt \right]
\]
\[ + \frac{1 + T_1(u) - \Psi(u) - T_2(u)}{1 + T_1(u)} \times \left[ \int_0^T \left( \frac{1}{2} \langle V'(u), u \rangle - V(u + z) \right) \, dt \right]
\]
\[ + \frac{1}{2} \left( \frac{T_2(u) + T_1(u)}{1 + T_1(u)} \right) \int_0^T \langle V'(u), u \rangle \, dt
\]
Now $0 \leq \mathcal{P}(u) \leq 1$ and as in Rabinowitz [R] one can show that $T_1(u)$, $T_2(u) \to 0$ as $M_0 \to +\infty$, so one can estimate

$$I(u) - \frac{\langle J'(u), u \rangle}{2(1 + T_1(u))} \geq \left( \frac{\mathcal{P}(u) + T_2(u)}{1 + T_1(u)} \right) \left( \frac{1}{2} - \frac{1}{p} \right) \int_0^T \langle V(u + z), a_4 \rangle dt$$

$$+ \left( \frac{1 + T_1(u) - \mathcal{P}(u) - T_2(u)}{1 + T_1(u)} \right) \left( \frac{1}{2} - \frac{1}{p} \right) \left[ V(u) + a_4 \right] dt - a_{13} \int_0^T |u|^q dt$$

$$+ \frac{1}{2} \left[ T_3(u) - T_1(u) \right] \int_0^T \langle V'(u), u \rangle dt$$

$$\geq \left( \frac{1}{2} - \frac{1}{p} \right) \int_0^T \langle V(u + z), a_4 \rangle dt - a_{14} \int_0^T |u|^q dt$$

$$+ \frac{1}{2} \left[ T_3(u) - T_1(u) \right] \int_0^T \langle V'(u), u \rangle dt. \quad (25)$$
Now since $T_1(u)$, $T_2(u) \to 0$ as $M \to +\infty$ the coefficient of the last term on the right can be taken as small as we wish. Comparing the right hand sides of (25) and (23), one then gets
\[
\int_0^T \left[ V(u+z) + a_4 \right] dt \leq 2A(I(u)^2 + 1)^{1/2},
\]
which implies $3^0$. The proof of $4^0$ is similar and is left to the reader. \[\]

Now, we recall the following variational principle of Rabinowitz [R] which is the key to the perturbation method.

**Theorem 5.** Suppose $J \in C^1(H)$ satisfies the $(P-S)$ condition. Let $W \subset H$ be a finite dimensional subspace of $H$, $\omega^* \in H \setminus W$ and let $W^* = W \oplus \text{span}\{\omega^*\}$. Also let
\[
W^*_+ = \{ \omega + t\omega^*, \omega \in W, t \geq 0 \}
\]
denote the upper half space in $W$. Suppose:

(i) $J(0) = 0$
(ii) $\exists R > 0$ such that $u \in W, \|u\| \geq R \Rightarrow J(u) \leq 0$
(iii) $\exists R^* > R$ such that $u \in W^*, \|u\| \geq R^* \Rightarrow J(u) \leq 0$. Let
\[
\Gamma = \{ h \in C^0(H, H), h \text{ odd, } h(u) = u \text{ if } \max(J(u), J(-u)) \leq 0 \}
\]
If
\[
\beta^* = \inf_{h \in \Gamma} \sup_{u \in W^*_+} J(h(u)) > \beta = \inf_{h \in \Gamma} \sup_{u \in W} J(h(u)) \geq 0,
\]
then, the functional $J$ possesses a critical value that is larger than $\beta^*$.

For a proof, we refer to Struwe [S]. Now we are ready to complete the proof of Theorem 1.

**Proof of Theorem 1.** We use the notations used in the proof of Lemma 2. We take $J$ as in (21), and define
\[
l_n = \inf_{h \in \Gamma} \sup_{u \in E_n^-} J(h(u))
\]
where $E_n^-$ is defined in the proof of Lemma 2 and
\[
\bar{\Gamma} = \{ h \in C^0(H, H), h \text{ is odd, } h(u) = u \text{ if } \max(J(u), J(-u)) \leq 0 \}.
\]
Clearly
\[ l_n \geq \beta_1 d_n - \beta_2 \quad \beta_1, \beta_2 > 0, \quad \beta_i \text{ independent of } n. \]

In view of (20), we therefore get

\[ l_n \geq \beta_3 n^{\frac{2p}{p-2}} \quad \text{for } \text{n large}, \quad \beta_3 > 0 \quad \text{independent of } n. \tag{26} \]

By Theorem 5, \( J \) and consequently by 3° of Theorem 4, \( I \) has a sequence of critical values going to infinity unless

\[ \inf_{h \in \Gamma_u(E_n)} \sup_{u \in E_n} \langle h(u) \rangle = \inf_{h \in \Gamma_u(E_n)} \sup_{u \in E_n} J(h(u)) = l_n. \]

But, since \( J \) satisfies 2° of Theorem 4, this implies

\[ l_n + 1 \leq l_n + \beta_4 (l_n^{p-1}/n + 1) \quad \beta_4 \text{ independent of } n, \tag{27} \]

which in turn gives the following estimate on \((l_n)_{n \geq 1}\)

\[ l_n \leq \beta_3 n^p \quad \beta_3 \text{ independent of } n. \tag{28} \]

This clearly contradicts (26) whenever \( p < 4 \) and the proof of Theorem 1 is now complete.

REFERENCES


Solution for a classical problem in the calculus of variations via rationalized Haar functions. Kybernetika, Vol. 37 (2001), No. 5, [575]–583. There has been a considerable renewal of interest in the classical problems of the calculus of variations both from the point of view of mathematics and of applications in physics, engineering, and applied mathematics. Finding the brachistochrone, or path of quickest decent, is a historically interesting problem that is discussed in virtually all textbooks dealing with the calculus of variations. In 1696, the brachistochrone problem was posed as a challenge to mathematicians by John Bernoulli. The solution of the brachistochrone problem is often cited as the origin of the calculus of variations. It can be shown that a nonlinear differential for the shape $y(x)$ of the path is $y[1 + (y')^2] = k$, where $k$ is a constant. First solve for $dx$ in terms of $y$ and $dy$, and then use the substitution $y = k \sin^2 \theta$ to obtain a parametric form of the solution. The curve turns out to be a cycloid. Students also viewed these Mathematics questions. Suppose an RC-series circuit has a variable resistor. If the resistance at time $t$ is defined by $R(t) = k_1 + k_2 t$, where $k_1$ and $k_2$ are known positive constants, then the differential equation in (9) of Section 3.1 becomes $\frac{d^2 x}{dt^2} + \frac{k_1}{C} \frac{dx}{dt} + \frac{k_2}{C} x = 0$. In the treatment of cancer of